[Prof. Johnson says that his intention was, in proposing problem 33, to require an integral equation between x and y referred to rectangular axes. The special interest, he remarks, in the problem consists in the avoidance of radicals which have not properly the double sign; and he requests us to propose the problem of finding the rectangular coordinates of the double point not on the axis of x, referring to Mr. Stille's figure in No. 9. Ed.]

FOLIATE CURVES.

BY PROF. E. W, HYDE, ITHACA, N. Y.

Prop. The foliate curves represented by the equatiou $\rho = a\cos n\theta$ (or $\rho = a\sin n\theta$) are hypotrochoids if n* be an integer, and both hypotrochoids and epitroehoids if n be fractional.

1. The equations of the hypotrochoid are

$$(1)\qquad x = (r_1 - r_2)\cos\psi + mr_2\cos\left(\frac{r_1 - r_2}{r_2}.\psi\right)$$

$$(2)\qquad y = (r_1 - r_2)\sin\psi - mr_2\sin\left(\frac{r_1 - r_2}{r_2}.\psi\right),$$

in which r_1 = radius of fixed circle,
r_2 = " " rolling " ,
mr_2 = distance of generating point from center of rolling circle, and ψ = angle between the axis of x and the radius of the rolling circle containing the generating point.

Let $r_1 = pr_2 = \dfrac{pa}{2(p-1)}$, and $m = p - 1$.

Substituting in (1) and (2) we have

$$(3)\qquad x = \tfrac{1}{2}a[\cos\psi + \cos(p-1).\psi] = a\cos(\tfrac{1}{2}p.\psi)\cos[\tfrac{1}{2}(2-p).\psi],$$

$$(4)\qquad y = \tfrac{1}{2}a[\sin\psi - \sin(p-1).\psi] = a\cos(\tfrac{1}{2}p.\psi)\sin[\tfrac{1}{2}(2-p).\psi].$$

Divide (4) by (3) and we get

$$(5)\qquad \frac{y}{x} = \frac{\sin[\frac{1}{2}(2-p).\psi]}{\cos[\frac{1}{2}(2-p).\psi]} = \tan[\tfrac{1}{2}(2-p).\psi] = \tan\theta,$$ where θ equals the angle between ρ and the axis of x.

$\therefore\ \tfrac{1}{2}(2-p).\psi = \theta$, whence $\psi = \dfrac{2\theta}{2-p}$ and $\tfrac{1}{2}p\psi = \dfrac{p\theta}{2-p}$.

Substituting these values of ψ in equation (3)

$$x = \rho\cos\theta = a\cos\frac{p\theta}{2-p}\,.\cos\theta,$$

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* Since n, m, and p are merely numerical muitipliers they are intrinsically positive.

(6) $\therefore \rho = a \cos \dfrac{p\theta}{2-q}$,

(7) $\therefore \rho = a \cos n\theta$ where $n = \dfrac{p}{2-p}$.

We have $p = \dfrac{2n}{n+1}$ and $a = \dfrac{2r_1(p-1)}{p} = \dfrac{r_1(n-1)}{n}$,

from which the relations of r_1, r_2 and a can be found for any value of n.

E.g. If $n = 1$, $p = 1$, $\rho = a \cos \theta$, and $a = 0$,
" $n = 2$, $p = \frac{4}{3}$, $\rho = a \cos 2\theta$, " $a = \frac{1}{2}r_1$,
" $n = 3$, $p = \frac{3}{2}$, $\rho = a \cos 3\theta$, " $a = \frac{2}{3}r_1$; &c.

Since $\cos a = \cos(-a)$ we have also

(8) $\rho = a \cos \dfrac{p\theta}{p-2}$.

Whence $p = \dfrac{2n}{n-1}$ and $a = \dfrac{r_1(n+1)}{n}$.

$\therefore$ if $n = 1$, $p = \infty$, $\rho = a \cos \theta$, and $a = 2r_1$,
" $n = 2$, $p = 4$, $\rho = a \cos 2\theta$, " $a = \frac{3}{2}r_1$,
" $n = 3$, $p = 3$, $\rho = a \cos 3\theta$, " $a = \frac{4}{3}r_1$, &c.

Also from (6) and (7)
if $n = \frac{1}{2}$, $p = \frac{2}{3}$, $\rho = a \cos \frac{1}{2}\theta$, and $a = -r_1$,
" $n = \frac{1}{3}$, $p = \frac{1}{2}$, $\rho = a \cos \frac{1}{3}\theta$, " $a = -2r_1$, &c.

Thus the proposition is proved for hypotrochoids.

2. The equations of the epitrochoid are

(9) $x = (r_1 + r_2)\cos\psi - mr_2 \cos\left(\dfrac{r_1 + r_2}{r_2}.\psi\right)$,

(10) $y = (r_1 + r_2)\sin\psi - mr_2 \sin\left(\dfrac{r_1 + r_2}{r_2}\psi.\right)$,

and we have $r_1 = pr_2 = \dfrac{pa}{2(p+1)}$, and $m = p + 1$.

Substituting in (9) and (10)

(11) $x = \frac{1}{2}a[\cos\psi - \cos(p+1)\psi] = -a \sin\frac{1}{2}(p+2)\psi.\ \sin(-\frac{1}{2}p\psi)$,

(12) $y = \frac{1}{2}a[\sin\psi - \sin(p+1)\psi] = a \cos\frac{1}{2}(p+2)\psi.\ \sin(-\frac{1}{2}p\psi)$.

(13) $\therefore \dfrac{x}{y} = -\tan\frac{1}{2}(p+2)\psi = \cot[\frac{1}{2}\pi + \frac{1}{2}(p+2)\psi] = \cot\psi$,

$\therefore\ \pi + (p+2)\psi = 2\theta$ whence

$\psi = \dfrac{2\theta - \pi}{p+2} - \frac{1}{2}p\psi = \dfrac{d(\pi - 2\theta)}{2(p+2)}$ and $\frac{1}{2}(p+2)\psi = \frac{1}{2}(2\theta - \pi) = \theta - \frac{1}{2}\pi$.

Substituiing in (11) $x = \rho\cos\theta$

$$= -a \sin(\theta - \tfrac{1}{2}\pi)\sin\frac{p(\pi - 2\theta)}{2(p+2)} = a\cos\theta\sin(\tfrac{1}{2}\pi - \theta)\,\frac{p}{p+2}.$$

(14) $\therefore\ \rho = a \sin\left((\frac{1}{2}\pi - \theta)\frac{p}{p+2}\right)$. Let $\frac{1}{2}\pi - \theta = \theta'$

(15) $$\rho = a \sin\frac{p}{p+2}.\theta'.$$

In this θ' being the complement of θ is to be measured from the axis of y.

Since p is essentially positive, the coefficient $p \div (p + 2)$ must be less be less than unity, and hence n must be a proper fraction.

We have $p = \frac{2n}{1-n}$ and $a = \frac{2r_1(p+1)}{p} = \frac{r_1(n+1)}{n}$

$\therefore$ if $n = 1$, $p = \infty$, $\rho = a \sin \theta' = a \cos \theta$, and $a = 2r_1$,
" $n = \frac{1}{2}$, $p = 2$, $\rho = a \sin \frac{1}{2}\theta'$, " $a = 3r_1$,
" $n = \frac{1}{3}$, $p = 1$, $\rho = a \sin \frac{1}{3}\theta'$, " $a = 4r_1$,
&c., thus our demonstration is complete.

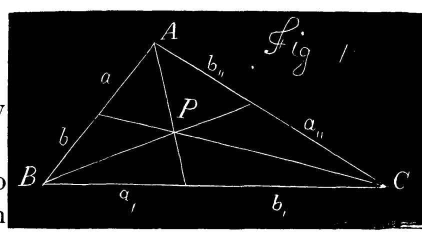
DETERMINATION OF ROOT OF N^th DEGREE.

BY DR. H. EGGERS, MILWAUKE, WISCONSIN.

1. I propose to give here an elementary method of extracting the root of any degree of a given number by an elementary geometrical process. This method is based on the following well known theorem of Ceva:

"A triangle ABC and an arbitrary point P in its plane are given. If we draw from the three vertices A, B, C, of the triangle three transversal lines through the point P, then on each side of the triangle two segments are formed: a and b; a_1 aud b_1; a_2 and b_2; which fulfil the relation:

$$\frac{a}{b} \cdot \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = -1"$$



This theorem solves immediately the problem:

" To constauct two lines, the ratio of which is the product of two given ratios"; which solution requires no explanation.

If we make the sides of the triangle equal, we can *square a given ratio*. If $a \div b$ is the given ratio, we construct an isosceles triangle with two sides equal to $a + b$. If now we draw two lines from the vertices A and C to the dividing points D and E, where AD = BE = a, and DB=EC=b, of the opposite sides, we find the point of intersection P, and the line connecting B with P marks on AC the point F, which solves the problem.